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Global and Local Quadratic Minimization ⋆

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Abstract. We present a method which when applied to certain non-convex QP will locate the global minimum, *all* isolated local minima and some of the non-isolated local minima. The method proceeds by formulating a (multi) parametric convex QP in terms of the data of the given non-convex QP. Based on the solution of the parametric QP, an unconstrained minimization problem is formulated. This problem is piece-wise quadratic. A key result is that the isolated local minimizers (including the global minimizer) of the original non-convex problem are in one-to-one correspondence with those of the derived unconstrained problem.

The theory is illustrated with several numerical examples. A numerical procedure is developed for a special class of non-convex QP's. It is applied to a problem from the literature and verifies a known global optimum and in addition, locates a previously unknown local minimum.

Key words: Global optimization, parametric quadratic programming, non-convex quadratic program.

1. Introduction

Here we consider the model non-convex quadratic programming problem

$$\text{QP} \qquad \min\{c'x + \tfrac{1}{2}x'Cx + x'DQ'x \mid Ax \le b\},$$

where $c \in \mathrm{E}^n$, $b \in \mathrm{E}^m$, A is an $m \times n$-matrix, D and Q are $n \times k$-matrices, C is a symmetric $n \times n$ positive semi-definite matrix, $k < n$ and $x \in \mathrm{E}^n$ is a variable. Corresponding to QP, we consider the parametric quadratic program:

$$\text{QP}(t) \qquad \min\{c'x + \tfrac{1}{2}x'Cx + t'Q'x \mid Ax \le b, D'x = t\},$$

where t is a parameter in E^k. Let R and $R(t)$ be feasible regions for QP and QP(t), respectively. Let arg min$\{$QP$(t)\}$ denote the set of all optimal solutions for QP(t). Finally, we formulate

$$\text{NP} \qquad \min\{f(t) \mid t \in \mathrm{E}^k\},$$

where

$$f(t) = \begin{cases} \inf\{c'x + \frac{1}{2}x'Cx + t'Q'x \mid x \in R(t)\}, & \text{if } R(t) \ne \phi, \\ +\infty, & \text{otherwise.} \end{cases}$$

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The non-convexity of the objective function of QP stems from the term $x'DQ'x$. One might question the generality of this model and suggest that the term be written as $x'Hx$, where H is a more general symmetric matrix, perhaps satisfying some properties. This situation has been analyzed by the authors in a companion paper, Best and Ding (1996), where it is shown that for any symmetric matrix H having full rank, there exist $n \times k$-matrices Q and D satisfying $H = \frac{1}{2}[DQ' + QD']$ (and so $x'Hx = x'DQ'x$) if and only if H has at least two nonzero eigenvalues of opposite sign. In addition, when the required condition is satisfied a method to construct such D and Q is given. For the purposes of this paper, we will assume that D and Q are already available.

We note that, in general, the problem of checking isolated local optimality is NP-hard, See Murty and Kabadi (1987), and, Pardalos and Schnitger (1988).

We will organize this paper as follows. In Section 2, we will develop the relationships between QP, QP(t) and NP. In particular, we will establish the one-to-one correspondence between isolated local minimizers of QP and NP. In Section 3, we will specialize these results to the class of non-convex quadratic programs with a Hessian which has exactly one negative eigenvalue. We will give an algorithm that can not only find a global minimizer, but can also find all isolated minimizers and some non-isolated local minimizers.

2. The Relationships between QP, QP(t) and NP

We begin this section with a small example problem which will illustrate the critical relationship between QP and NP.

EXAMPLE 2.1.

$$\text{QP} \qquad \begin{aligned} \text{minimize}: \;& x_1x_2 \\ \text{subject to}: \;& x_1 \geq 0.5, \quad 22x_1 + 8x_2 \geq 27, \\ & 8x_1 + 22x_2 \geq 27, \quad x_2 \geq 0.5. \end{aligned}$$

Here, $C = 0$, $n = 2$ and we may take $D = (1, 0)'$ and $Q = (0, 1)'$. QP(t) can be written as

$$\text{QP}(t) \qquad \begin{aligned} \text{minimize}: \;& tx_2 \\ \text{subject to}: \;& x_1 \geq 0.5, \quad 22x_1 + 8x_2 \geq 27, \\ & 8x_1 + 22x_2 \geq 27, \quad x_2 \geq 0.5, \\ & x_1 = t. \end{aligned}$$

The solution of QP(t) is a piece-wise linear function of t and is summarized in Table 2.1.

Examination of QP(t) with Table 2.1 gives $f(t)$:

$$f(t) = \begin{cases} t(27 - 22t)/8, & \text{if } 0.5 \leq t \leq 0.9, \\ t(27 - 8t)/22, & \text{if } 0.9 \leq t \leq 2, \\ t/2, & \text{if } t \geq 2. \end{cases}$$

Table 2.1. Optimal Solution for QP(t) for Example 2.1.

$t < 0.5$	$0.5 \leq t \leq 0.9$	$0.9 \leq t \leq 2.$	$t \geq 2.$
no feasible solution	$\begin{bmatrix} 0 \\ \frac{27}{8} \end{bmatrix} + t \begin{bmatrix} 1 \\ -\frac{22}{8} \end{bmatrix}$	$\begin{bmatrix} 0 \\ \frac{27}{22} \end{bmatrix} + t \begin{bmatrix} 1 \\ -\frac{8}{22} \end{bmatrix}$	$\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

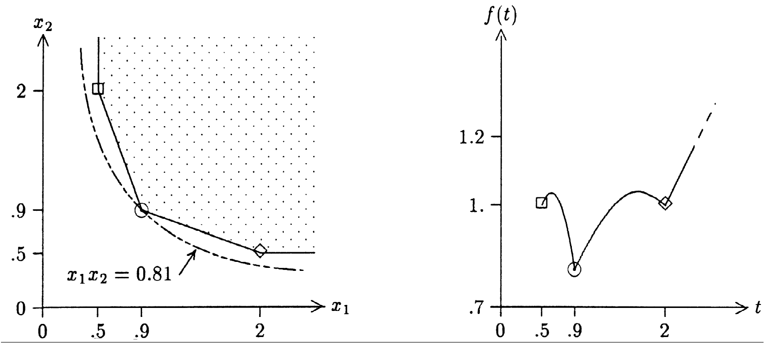

Figure 2.1. (a) Example 2.1. (b) $f(t)$ for Example 2.1.

Example 2.1 is illustrated in Figures 2.1(a) and (b). Figure 2.1(a) shows the given non-convex problem. The feasible region is shown as the shaded area. The level set $x_1x_2 = 0.81$ is shown with a broken line. It is clear from the figure that there are local minima at $(0.5, 2)'$ and $(2, 0.5)'$ and the global minimum occurs at $(0.9, 0.9)'$. Figure 2.1(b) shows $f(t)$, a piece-wise quadratic function which by inspection, has isolated local minima at $t = 0.5$, and 2 and a global minimum at $t = 0.9$. Using Table 2.1, we see that arg min$\{$QP$(0.5)\} = (0.5, 2)'$, arg min$\{$QP$(2)\} = (2, 0.5)'$ and arg min$\{$QP$(0.9)\} = (0.9, 0.9)'$. Thus, the local (global) minima of QP and $f(t)$ are in one-to-one correspondence for this example. Also note that $f(t)$ is a piece-wise quadratic function of a single variable and so it is straightforward to obtain its local and global minimizers.

Notice that in Example 2.1, the local minimizers for both QP and NP are isolated. The requirement that the local minimizers of QP be isolated is key in obtaining the one-to-one correspondence between such points of QP and NP. The final result will be formulated in Theorem 2.4 and will be a consequence of Theorems 2.1 - 2.3, following.

THEOREM 2.1. *Let t^* be a local minimizer for NP with $f(t^*) > -\infty$. Then any $x^* \in$ arg min$\{$QP$(t^*)\}$ is a local minimizer for QP.*

Proof. Since t^* is a local minimizer for NP, there exists a $\delta > 0$ such that

$$f(t) \geq f(t^*) \quad \text{for any } t \in B_\delta(t^*), \tag{2.1}$$

where $B_\delta(t^*) = \{t \in \mathrm{E}^k \mid \|t - t^*\| \le \delta\}$. Now assume to the contrary, that there is an $x^* \in \arg\min\{\mathrm{QP}(t^*)\}$ which is not a local minimizer for QP. Then there exists a sequence $\{x^i\}$ such that

$$c'x^i + \tfrac{1}{2}(x^i)'Cx^i + (x^i)'DQ'x^i < c'x^* + \tfrac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^*,$$

where $Ax^i \le b$, and $x^i \to x^*$. Since $x^* \in \arg\min\{\mathrm{QP}(t^*)\}$,

$$f(t^*) = c'x^* + \tfrac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^*.$$

Hence

$$c'x^i + \tfrac{1}{2}(x^i)'Cx^i + (x^i)'DQ'x^i < f(t^*). \tag{2.2}$$

Since $x^i \to x^*$, $D'x^i \to D'x^* = t^*$. Thus, there is an $M > 0$ such that

$$D'x^i \in B_\delta(t^*), \quad \text{whenever } i > M. \tag{2.3}$$

Let $t^i = D'x^i$ for $i > M$. Then

$$\begin{aligned} f(t^i) &= \inf\{c'x + \tfrac{1}{2}x'Cx + (t^i)'Q'x \mid Ax \le b, D'x = t^i\} \\ &\le c'x^i + \tfrac{1}{2}(x^i)'Cx^i + (t^i)'Q'x^i \qquad \text{(since } Ax^i \le b, D'x^i = t^i) \\ &= c'x^i + \tfrac{1}{2}(x^i)'Cx^i + (x^i)'DQ'x^i < f(t^*) \qquad \text{(from (2.2))}. \end{aligned}$$

But from (2.1) and (2.3), we have $f(t^i) \ge f(t^*)$, a contradiction. The assumption that there is an $x^* \in \arg\min\{\mathrm{QP}(t^*)\}$ which is not a local minimizer for QP leads to a contradiction and is therefore false. The proof of the theorem is thus complete. □

If t^* is an isolated local minimizer of f on E^k, we have the following further result.

THEOREM 2.2. *If t^* is an isolated local minimizer for f on E^k with $f(t^*) > -\infty$ and* arg min$\{\mathrm{QP}(t^*)\}$ *is the singleton point $\{x^*\}$, then x^* is an isolated local minimizer for QP.*

Proof. From Theorem 2.1, x^* is a local minimizer for QP. If x^* is not an isolated local minimizer for QP, there exist a sequence $\{x^i\} \subset R$, $x^i \to x^*$ and $x^i \ne x^*$ for all i such that $c'x^i + \frac{1}{2}(x^i)'Cx^i + (x^i)'DQ'x^i = c'x^* + \frac{1}{2}(x^*)'Cx^* + (x^*)'DQ'x^*$. Let $t^i = D'x^i$. Then $t^i \to t^*$ and $f(t^i) \le f(t^*)$. Since arg min$\{\mathrm{QP}(t^*)\} = \{x^*\}$ and $x^i \ne x^*$ for all i, $t^i \ne t^*$ for all i. This contradicts that t^* is an isolated local minimizer for NP. The proof of the theorem is thus complete. □

Theorem 2.2 is illustrated in Example 2.1 where each of the three local minimizers for $f(t)$ are isolated, their corresponding sets, arg min$\{\mathrm{QP}(t)\}$, are singletons and each such point is an isolated local minimizer for QP. The following example illustrates Theorem 2.1 and in addition, shows that the condition arg min$\{\mathrm{QP}(t^*)\}$ be a single point cannot, in general, be removed from Theorem 2.2.

EXAMPLE 2.2.

QP $\qquad\qquad \min\{-\frac{1}{4}x_1 - \frac{1}{2}x_2 + x_1x_2 \mid x_1 \geq \frac{1}{2}, \frac{3}{8} \leq x_2 \leq \frac{1}{2}\}.$

Here we take $C = 0$, $n = 2$, $D = (1,0)'$ and $Q = (0,1)'$. Then $D'x = x_1$, $Q'x = x_2$ and QP(t) becomes

QP(t) $\qquad\qquad \min\{-\dfrac{1}{4}t + (t - \frac{1}{2})x_2) \mid \ t \geq \frac{1}{2},\ \frac{3}{8} \leq x_2 \leq \frac{1}{2}\ \},$

from which $f(t)$ is derived as:

$$f(t) = \begin{cases} \frac{1}{8}t - \frac{3}{16}, & \text{if } t \geq \frac{1}{2}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The situation is illustrated in Figures 2.2(a) and 2.2(b). The feasible region for QP is shown as the shaded area in Figure 2.2(a). It is clear that $t^* = \frac{1}{2}$ is an isolated local minimizer for f on E^1. Indeed, it is also the global minimizer. See Figure 2.2(b). However, arg min$\{$QP$(t^*)\} = \{(\frac{1}{2}, x_2)' \mid \frac{3}{8} \leq x_2 \leq \frac{1}{2}\}$ and by Theorem 2.1, each one of these points is a local (indeed, global) minimizer for QP. These are shown by the darkened line in Figure 2.2(a). Clearly, none of the local minimizers for QP are isolated. Thus the condition arg min$\{$QP$(t^*)\}$ be a singleton is necessary in Theorem 2.2.

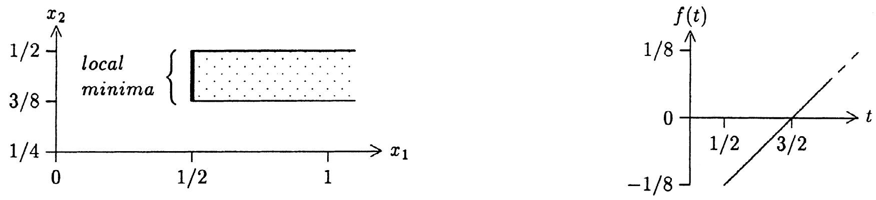

Figure 2.2. (a) Example 2.2. (b) $f(t)$ for Example 2.2.

EXAMPLE 2.3.

QP $\qquad\qquad \min\{-\frac{1}{4}x_1 - \frac{1}{2}x_2 + x_1x_2 \mid x_1 \geq \frac{1}{2},\ 0 \leq x_2 \leq \frac{1}{2}\}.$

Here we take $C = 0$, $n = 2$, $D = (1,0)'$ and $Q = (0,1)'$, Then $D'x = x_1$, $Q'x = x_2$ and QP(t) becomes

QP(t) $\qquad\qquad \min\{-\frac{1}{4}t \ + \ (t \ - \ \frac{1}{2})x_2 \mid t \geq \frac{1}{2},\ 0 \leq x_2 \leq \frac{1}{2}\},$

from which $f(t)$ is derived as:

$$f(t) = \begin{cases} -\frac{1}{4}t, & \text{if } t \geq \frac{1}{2}, \\ +\infty, & \text{otherwise.} \end{cases}$$

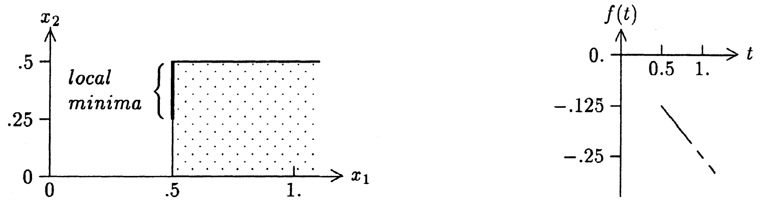

Figure 2.3. (a) Example 2.3. (b) $f(t)$ for Example 2.3.

The situation is illustrated in Figures 2.3(a) and 2.3(b).

Observe that

$$\arg\min\{\mathrm{QP}(\tfrac{1}{2})\} \;=\; \{\,(\tfrac{1}{2}, x_2)' \mid 0 \le x_2 \le \tfrac{1}{2}\},$$

and for $t > \frac{1}{2}$, arg min$\{\mathrm{QP}(t)\} \;=\; (t, 0)'$. Also observe that $\{(\frac{1}{2}, x_2)' \mid \frac{1}{4} < x_2 \le \frac{1}{2}\}$ are local optimal solutions for QP. In particular, $x^* = (\frac{1}{2}, \frac{1}{2})'$ is a local optimizer for QP but $t^* = D'x^* = \frac{1}{2}$ is *not* a local minimizer for f on E^1. Indeed f does not possess a local minimizer on E^1.

Example 2.3 shows that a one-to-one correspondence between local minima of QP and $f(t)$ will not hold without some restrictions. The key requirement in establishing the correspondence is that corresponding local minimizers for QP, QP(t) and $f(t)$ should each be isolated. This will be established subsequently. First we need the following lemma.

LEMMA 2.1. *Let x^* be an isolated local minimizer for QP and let $t^* = D'x^*$. Let $\{t^i\}$ be any sequence with $t^i \to t^*$ and let $x^i \in$* arg min$\{\mathrm{QP}(t^i)\}$. *If there exists an $M > 0$ such that $f(t^i) \le M$ for all i, then $\{x^i\}$ is bounded.*

Proof. Since x^* is an isolated local minimizer for QP, there exists a $\delta > 0$ such that

$$c'x^* + \tfrac{1}{2}(x^*)'Cx^* + (x^*)'DQ'x^* < c'x + \tfrac{1}{2}x'Cx + x'DQ'x \tag{2.4}$$

for any $x \in (B_\delta(x^*) \bigcap R) \setminus \{x^*\}$. So,

$$c'x^* + \tfrac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^* < c'x + \tfrac{1}{2}x'Cx + (t^*)'Q'x$$

for any $x \in (B_\delta(x^*) \bigcap R(t^*)) \setminus \{x^*\}$. Since $c'x + \frac{1}{2}x'Cx + (t^*)'Q'x$ is convex, we have

$$c'x^* + \tfrac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^* < c'x + \tfrac{1}{2}x'Cx + (t^*)'Q'x \tag{2.5}$$

for any $x \in R(t^*) \setminus \{x^*\}$. Hence arg min$\{\mathrm{QP}(t^*)\} = \{x^*\}$. This implies that there does not exist a nonzero vector s satisfying the following conditions

$$As \le 0,\; D's = 0, \tag{2.6}$$

$$(c + Qt^*)'s \le 0, \tag{2.7}$$

$$Cs = 0. \tag{2.8}$$

By Theorem 2.1 of Best and Ding (1995), f is lower semi-continuous at t^*. Therefore for any $\gamma > 0$ there exists an $\epsilon > 0$ such that $f(t) \geq f(t^*) - \gamma$ for any $t \in B_\epsilon(t^*)$. Since $t^i \to t^*$, there exists an $N > 0$ such that $t^i \in B_\epsilon(t^*)$ for all $i \geq N$. So,

$$f(t^*) - \gamma \leq f(t^i) \leq M \tag{2.9}$$

for all $i \geq N$. Now assume that on the contrary, $\{x^i\}$ is unbounded, then $\{x^i/\|x^i\|\}$ has a convergent subsequence. Without loss of generality, let

$$\lim_{i\to\infty} \frac{x^i}{\|x^i\|} = s \text{ and } \lim_{i\to\infty} \|x^i\| = +\infty. \tag{2.10}$$

From $f(t^i) = c'x^i + \frac{1}{2}(x^i)'Cx^i + (t^i)'Q'x^i$, (2.9) and (2.10), we have

$$\tfrac{1}{2}s'Cs = \lim_{i\to\infty} \tfrac{f(t^i)}{\|x^i\|^2} = 0$$

and

$$(c + Qt^*)'s + \lim_{i\to\infty} \frac{(x^i)'Cx^i}{2\|x^i\|} = \lim_{i\to\infty} \frac{f(t^i)}{\|x^i\|} = 0,$$

i.e.;

$$Cs = 0 \text{ and } (c + Qt^*)'s = -\lim_{i\to\infty} \frac{(x^i)'Cx^i}{2\|x^i\|} \leq 0.$$

From $Ax^i \leq b$ and $D'x^i = t^i$, we have $As \leq 0$ and $D's = 0$. Thus we have exhibited a non-zero s satisfying (2.6)-(2.8). This is a contradiction and the proof of the lemma is complete. □

THEOREM 2.3. *If x^* is an isolated local minimizer for QP, then $t^* = D'x^*$ is an isolated local minimizer for NP, $f(t^*) = c'x^* + \frac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^*$ and* arg min$\{$QP$(t^*)\} = \{x^*\}$.

Proof. As the proof of Lemma 2.1, we have (2.4), (2.5) and $f(t) \geq f(t^*) - \gamma$ for any $t \in B_\epsilon(t^*)$. Thus,

$$f(t^*) = c'x^* + \tfrac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^* \text{ and } \arg\min\{\mathrm{QP}(t^*)\} = \{x^*\}.$$

Now assume to the contrary, that t^* is not an isolated local minimizer for NP. Then there exist two sequences $\{x^i\}$ and $\{t^i\}$ with $t^i = D'x^i$, $t^i \to t^*$, $t^i \in B_\epsilon(t^*)$ and $x^i \in R \setminus B_\delta(x^*)$ such that

$$c'x^i + \tfrac{1}{2}(x^i)'Cx^i + (t^i)'Q'x^i = f(t^i) \leq f(t^*).$$

By Lemma 2.1, $\{x^i\}$ is bounded, so there exists a convergent subsequence. Without loss of generality, let $x^i \to x^0$. Then $x^0 \neq x^*$, $t^* = D'x^0$, $Ax^0 \leq b$ and

$$c'x^0 + \tfrac{1}{2}(x^0)'Cx^0 + (t^*)'Q'x^0 \leq f(t^*).$$

This contradicts (2.5). The proof of the theorem is thus complete. □

Combining Theorem 2.2 and Theorem 2.3, we have the following result.

THEOREM 2.4. *A point x^* is an isolated local minimizer for QP if and only if $t^* = D'x^*$ is an isolated local minimizer of NP, $f(t^*) = c'x^* + \frac{1}{2}(x^*)'Cx^* + (t^*)'Q'x^*$ and* arg min$\{$QP$(t^*)\} = \{x^*\}$.

REMARK 2.1. From Theorem 2.1 and Theorem 2.4, we know that f will keep all of the critical information concerning isolated local minimizers of QP and some of the local minimizers of QP. Thus if we can locate all local minimizers of f we will obtain all isolated local minimizers and some local minimizers of QP.

Although the one-to-one correspondence between local minimizers of QP and NP requires the condition of isolated local minima, this condition is not required for global minima as given in Theorem 2.5 below. The proof of this result can be obtained from definitions directly.

THEOREM 2.5. *A point $t^* \in \mathrm{E}^k$ with $f(t^*) > -\infty$ is a global minimizer of NP if and only if QP has a global minimizer x^* such that $D'x^* = t^*$, $f(t^*) = c'x^* + \frac{1}{2}(x^*)'\, Cx^* + (t^*)'Qx^*$.*

We complete this section by showing how to recognize whether a local minimizer is an isolated local minimizer. Suppose that we know t^* is an isolated local minimizer and we want to know whether corresponding point x^* is also an isolated local minimizer. In doing so, we need only verify that arg min$\{$QP$(t^*)\} = \{x^*\}$. If C is positive definite, then QP(t) is strictly convex. In this case, x^* is necessarily uniquely determined and consequently arg min$\{$QP$(t^*)\} = \{x^*\}$. Otherwise, since QP(t^*) is convex, we may assume that x^* is computed by some quadratic programming algorithm and $(u, v) = (u_1, \cdots, u_m, v_1, \cdots, v_k)$ is the associated vector of multipliers, where u and v correspond $Ax \leq b$ and $D'x = t^*$, respectively. Now by Theorem 4.14 of Best and Ritter (to appear), arg min$\{$QP$(t^*)\}$ can be represented by the set of x which satisfy

$$\left.\begin{array}{l} D'x = t^*,\ Cx = Cx^*, \\ a_i'x = b_i,\ \text{for all } i \text{ with } 1 \leq i \leq m,\ \text{and } u_i > 0, \\ a_i'x \leq b_i,\ \text{for all } i \text{ with } 1 \leq i \leq m,\ \text{and } u_i = 0. \end{array}\right\} \tag{2.11}$$

Let $I = \{i \mid 1 \leq i \leq m,\ a_i'x^* = b_i\}$ and A_I be a sub matrix of A induced by i-th row of A for $i \in I$. If rank$([D, C, A_I']) < n$, then arg min$\{$QP$(t^*)\}$ is not a singleton, i.e.; x^* is not an isolated local minimizer. In this case, an alternative local minimizer can be computed easily from the null space of $[D, C, A_I']'$. In fact, for any $y \in \mathrm{E}^n$, if $[D, C, A_I']'y = 0$ with $y \neq 0$, there is a nonzero number α such that $x^* + \alpha y$ satisfies (2.11), i.e.; $x^* + \alpha y$ is an alternate local minimizer. If rank$([D, C, A_I']) = n$, we need to consider the following linear programming problem:

$$\mu \ = \ \min \left\{ \sum_{i \in I} {a_i}' x \mid (2.11) \right\}.$$

If $\mu = \sum_{i \in I} b_i$ then arg min$\{$QP$(t^*)\}$ is a singleton, and x^* is an isolated local minimizer. Otherwise, x^* is not an isolated local minimizer and an optimal solution of the linear programming problem is an alternative local minimizer.

In next section, we are going to discuss some applications of the results established in this section.

3. The Case of a Single Negative Eigenvalue ($k = 1$)

In this section we consider our model problem with D and Q being n-dimensional *vectors*; i.e., $k = 1$. To emphasize this we replace D and Q with d and q, respectively. The model problem QP and the derived problems QP(t) and NP become

$$\text{QP}_1 \qquad \min\{c'x + \tfrac{1}{2}x'Cx + (d'x)(q'x) \mid Ax \leq b\},$$

where $c \in \mathrm{E}^n$, $b \in \mathrm{E}^m$, A is an $m \times n$-matrix, d and $q \in \mathrm{E}^n$, C is a symmetric $n \times n$ positive semi-definite matrix, and $x \in \mathrm{E}^n$. QP(t) becomes

$$\text{QP}_1(t) \qquad \min\{c'x + \tfrac{1}{2}x'Cx + tq'x \mid Ax \leq b, d'x = t\},$$

where t is a scalar parameter. Let R_1 and $R_1(t)$ be feasible regions for QP$_1$ and QP$_1(t)$, respectively. Let arg min$\{$QP$_1(t)\}$ denote the set of all optimal solutions for QP$_1(t)$. Finally, we formulate

$$\text{NP}_1 \qquad \min\{f_1(t) \mid t \in \mathrm{E}^1\},$$

where

$$f_1(t) = \begin{cases} \inf\{c'x + \frac{1}{2}x'Cx + tq'x \mid x \in R_1(t)\}, & \text{if } R_1(t) \neq \phi, \\ +\infty, & \text{otherwise,} \end{cases}$$

and we have used the subscript "1" throughout the above to emphasize that $k = 1$.

If QP$_1$ were written with a more general Hessian matrix H, rather than $C + \frac{1}{2}(dq' + qd')$ then the resulting problem could be transformed into one having a Hessian matrix of the latter form provided H had exactly one negative eigenvalue (hence the title of this section). Details of this transformation are given in Best and Ding (1996).

The problem QP$_1$ has been shown to be NP-hard by Pardalos and Vavasis (1991). Konno *et al.* (1991) proposed a solution method for a variation of QP$_1$ for which the constraints were equalities and non-negativity constraints. The method used a parametric form of the simplex algorithm and was designed solely to find a global minimizer. In this section, we will also develop a method to solve QP_1. However, in contrast to the method of Konno, our method will locate a global minimizer (if one exists), all isolated local minimizers and some non-isolated local minimizers. Indeed, even if QP_1 does not possess a global minimizer our method will locate all isolated local minimizers and some non-isolated local minimizers. Moreover, we will show that isolated local minimizers can be distinguished from non-isolated local minimizers by solving a linear programming problem.

Note that $\mathrm{QP}_1(t)$ is a convex parametric quadratic programming problem, with the parameter being a scalar. Note also that the parameter t occurs in both the linear part of the objective function as well as the right hand-side of a constraint. $\mathrm{QP}_1(t)$ must be solved for all possible t. An appropriate method to use is that of Best (1996). Best's method allows explicitly for a parameter in both the linear part of the objective function and the right hand-side of the constraints. Also, it allows for the possibility that the Hessian of the parametric QP is positive semi-definite, rather than just positive definite. In addition, it supplies critical information as to the status of $\mathrm{QP}_1(t)$ at the end points of the parametric interval.

Applying Best's method to $\mathrm{QP}_1(t)$ will produce numbers $t_0, t_1, \cdots, t_\nu$ and n-vectors $h_{0i},\ h_{1i},\ i = 1, \ldots, \nu$ satisfying

$$x_i(t) \ = \ h_{0i} \ + \ t h_{1i} \tag{3.1}$$

is optimal for $\mathrm{QP}_1(t)$ for all t with $t_{i-1} \le t \le t_i$ and for all $i = 1, \ldots \nu$. It is possible to have $t_0 = -\infty$ and/or $t_\nu = +\infty$. If $t_0 > -\infty$, Best's method will conclude that $\mathrm{QP}_1(t)$ is either unbounded from below or infeasible for $t < t_0$, and, the relevant possibility will be given. Similarly, if $t_\nu < \infty$, then the method will conclude that $\mathrm{QP}_1(t)$ is either unbounded from below or has no feasible solution for $t > t_\nu$ and the relevant possibility will be stated. Table 2.1 gives the relevant information for Example 2.1.

Having solved $\mathrm{QP}_1(t)$, it remains to solve NP_1. Using h_{0i} and h_{1i} from (3.1), define the constants

$$\left.\begin{aligned} \gamma_{1i} &= c' h_{0i} + \tfrac{1}{2} h_{0i}{}' C h_{0i}, \\ \gamma_{2i} &= c' h_{1i} + h_{0i}{}' C h_{1i} + q' h_{0i}, \\ \gamma_{3i} &= \tfrac{1}{2} h_{1i}{}' C h_{1i} + q' h_{1i}, \end{aligned}\right\} \tag{3.2}$$

for $i = 1, \ldots, \nu$. From (3.1), (3.2) and the definition of $f(t)$, we now have

$$f_1(t) \ = \ \begin{cases} \gamma_{11} + \gamma_{21} t + \gamma_{31} t^2, & \text{if } t_0 \le t \le t_1, \\ \gamma_{12} + \gamma_{22} t + \gamma_{32} t^2, & \text{if } t_1 \le t \le t_2, \\ \quad \cdots & \quad \cdots \\ \gamma_{1\nu} + \gamma_{2\nu} t + \gamma_{3\nu} t^2, & \text{if } t_{\nu-1} \le t \le t_\nu. \end{cases} \tag{3.3}$$

This shows that $f_1(t)$ is piece-wise quadratic on ν adjacent intervals. This is illustrated in Figure 2.1(b) with $\nu = 3, t_0 = 0.5,\ t_1 = 0.9,\ t_2 = 2.$, and $t_3 =$

Table 3.1. Determination of Local Minimizers for $f_1(t)$

Case	Range of i	Conditions	Local Min of $f(t)$
1	$1 \le i \le \nu - 1$	$\gamma_{2i} + 2\gamma_{3i}t_i \le 0$, and, $\gamma_{2,i+1} + 2\gamma_{3,i+1}t_i \ge 0$	t_i
2	$1 \le i \le \nu$	$\gamma_{3i} > 0$, and, $t_{i-1} < -\gamma_{2i}/(2\gamma_{3i}) < t_i$	$-\gamma_{2i}/(2\gamma_{3i})$
3		$t_0 > -\infty$, $\gamma_{20} + 2\gamma_{30}t_0 > 0$ and $R_1(t) = \emptyset$ for $t < t_0$	t_0
4		$t_\nu < \infty$, $\gamma_{2\nu} + 2\gamma_{3\nu}t_\nu < 0$ and $R_1(t) = \emptyset$ for $t > t_\nu$	t_ν
5	$1 \le i \le \nu$	$\gamma_{2i} = \gamma_{3i} = 0$	$t_{i-1} < t < t_i$

∞. The simple nature of $f(t)$ allows the determination of its local minima as summarized in Table 3.1.

Case 1 concerns points where the left derivative of $f_1(t)$ is negative and the right derivative is positive. This possibility is illustrated in Figure 2.1(b) with $t_1 = .9$ and $t_2 = 2$. Case 2 corresponds to $f_1(t)$ being strictly convex on $[t_{i-1}, t_i]$ and the unconstrained minimum of that quadratic piece lying within the interval. Case 3 requires that t_0 be finite, $f(t)$ be increasing at t_0 and that there be no feasible solutions below t_0. This is illustrated in Figure 2.1(b) for $t_0 = 0.5$. Note that the relevant possibility will be given by Best's parametric QP method. Also note that if the QP algorithm determines that $\mathrm{QP}_1(t)$ is unbounded from below for $t < t_0$, then t_0 is *not* a local minimizer for $f_1(t)$. For the right-hand end of the interval, Case 4 is analogous to Case 3. Case 5 occurs when $f(t)$ is constant on the open interval (t_{i-1}, t_i), in which case any point in the interval is a local minimizer. The end points of the interval may or may not be local minimizers. See the discussion following Theorem 3.1.

The following result is an immediate consequence of Theorem 2.1

THEOREM 3.1. *Let $t_1^*, t_2^*, \cdots, t_N^*$ be obtained from (3.1), (3.2) and Table 3.1. Let $x_i^* \in \arg\min\{\mathrm{QP}(t_i^*)\}$ for $i = 1, \cdots, N$. Then $x_i^*, i = 1, \cdots, N$ are all local minimizers of* QP_1. *Moreover, if* QP_1 *possesses a global minimizer, then it is that x_k^* which gives the smallest objective function value for* QP_1 *among all the* $\{x_i^* \mid i = 1, \cdots, N\}$.

The formulation of Theorem 3.1 does not explicitly allow for Case 5 of Table 3.1. because it deals with particular points rather than points and intervals. If Case

5 does apply, then arg min$\{\text{QP}(t)\}$ are all local minimizers of QP_1. If the left derivative of $f(t)$ is negative at t_{i-1} then t_{i-1} is also a local minimizer of $f(t)$ and consequently all elements of arg min$\{\text{QP}(t_{i-1})\}$ are local minimizers of QP_1. The analogous result holds for the right-hand end of the interval.

The information concerning whether QP_1 possesses a global minimizer can be obtained from Best's algorithm, (3.1), (3.2) and Table 3.1. This can be summarized as follows. When Best's algorithm terminates with a finite t_0, it also specifies that either $\text{QP}_1(t)$ is unbounded from below for $t < t_0$, or, $R_1(t) = \phi$ for $t < t_0$. The analogous result holds when t_ν is finite. Thus if $R_1(t) \neq \phi$ for $t < t_0$ with t_0 being finite or $R_1(t) \neq \phi$ for $t > t_\nu$ with t_ν being finite, then QP_1 has no global minimizer. Otherwise both

$$\min \{\gamma_{11} + \gamma_{21}t + \gamma_{31}t^2 \mid t_0 \leq t \leq t_1\}$$

and

$$\min \{\gamma_{1\nu} + \gamma_{2\nu}t + \gamma_{3\nu}t^2 \mid t_{\nu-1} \leq t \leq t_\nu\}$$

have global minimizers if and only if QP_1 has global minimizer.

We illustrate this procedure by applying it to an example from Floudas and Pardalos (1990).

EXAMPLE 3.1.

$$\text{QP}_1 \qquad \begin{array}{rl} \text{minimize}: & 6.5x - 0.5x^2 - y_1 - 2y_2 - 3y_3 - 2y_4 - y_5 \\ \text{subject to}: & AX \leq b,\ 0 \leq X = (x, y)',\ y_i \leq 1,\ i = 3, 4, \\ & y_5 \leq 2, x \in \text{E}^1,\ y \in \text{E}^5, \end{array}$$

where

$$b = \begin{bmatrix} 16 \\ -1 \\ 24 \\ 12 \\ 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 & 8 & 1 & 3 & 5 \\ -8 & -4 & -2 & 2 & 4 & -1 \\ 2 & 0.5 & 0.2 & -3 & -1 & -4 \\ 0.2 & 2 & 0.1 & -4 & 2 & 2 \\ -0.1 & -0.5 & 2 & 5 & -5 & 3 \end{bmatrix}.$$

This problem has a known global minimizer $(x^*, y^*) = (0, 6, 0, 1, 1, 0)'$ with optimal objective function value of -11.

Application of our algorithm to this problem confirms that the above solution is indeed the global optimum. In addition, it also determines that the global minimizer is isolated and that $(\bar{x}, \bar{y}) = (13.83, 0, 0, 1, 0.19, 0.12)'$ is an isolated local minimizer with objective function value -9.26. That is, the problem not only has an isolated global minimizer, but also a previously undiscovered isolated local minimizer.

In order to further test our algorithm, we formulated some variations of this problem. In all cases, the constraints remained the same and only the linear part of

Table 3.2. Objective Functions for Test Problems

$g_1(x,y) =$	$-0.5x^2 + 6.5x - y_1 - 2y_2 - 3y_3 - 2y_4 - y_5$
$g_2(x,y) =$	$-0.5x^2 + 6.5x - 2y_2 - 3y_3 - y_5 + 2xy_1 + 3xy_2$ $-5xy_3 - 4xy_4 + 6xy_5$
$g_3(x,y) =$	$-0.5x^2 + 6.5x - 4y_2 - 4y_3 - y_5 + 2xy_1 + 3xy_2$ $-5xy_3 - 4xy_4 + 6xy_5$
$g_4(x,y) =$	$-0.5x^2 + 6.5x + 2xy_1 + 3xy_2 + 5xy_3 - 4xy_4 + 6xy_5$

Table 3.3. Local and Global Minima for Four Test Problems

Objective Function	Objective Value	Solution Points	Type of Minimum
$g_1(x,y)$	-11	$(0, 6, 0, 1, 1, 0)'$	global min, isolated
	-9.2567	$(13.83, 0, 0, 1, 0.19, 0.12)'$	local min, isolated
$g_2(x,y)$	-105	$(12, 0, 0, 1, 1, 0)'$	global min, isolated
	-5.6583	$(0, 0.92, 1.33, 1, 0.84, 0)'$	local min, isolated
	-5.0718	$(0.52, 0, 1.44, 1, 1, 0)'$	local min, isolated
$g_3(x,y)$	-106	$(12, 0, 0, 1, 1, 0)'$	global min, isolated
	-9.3166	$(0, 0.92, 1.33, 1, 0.84, 0)'$	local min, isolated
	-8.9409	$(0.5, 0, 1.45, 1, 0.97, 0)'$	local min, isolated
	-8.9428	$(0.52, 0, 1.44, 1, 1, 0)'$	local min, isolated
$g_4(x,y)$	-46.875	$(12.5, 0, 0, 0, 1, 0)'$	global min, isolated
	1.3672	$(0.625, 0, 0, 0, 1, 0)'$	local min, isolated
	0	$(0, 1.25, 0, 0, 1, 0)'$	local min, non-isolated
	0	$(0, 7.6, 0, 0.8, 0, 0)'$	local min, non-isolated

the objective function was changed. The modified objective functions, $g_i(x,y),\ i = 1,\ldots 4$ are shown in Table 3.2 along with their corresponding vectors d and q. The original Floudas and Pardalos problem corresponds to $g_1(x,y)$.

Each of the four examples was solved with $d = (1,0,0,0,0,0)'$. The first example used $q = (-0.5,0,0,0,0,0)'$ and the remaining three used $q = (-0.5,2,3,-5,-4,6)'$. The results of applying our method to these problems are summarized in Table 3.3. Note that the results summarized in Table 3.3 show that our method located two non-isolated local minimizers for the fourth test problem. This shows that although we cannot guarantee that our method will find all non-isolated local minimizers, it still may find some, or even all.

4. Conclusions

We have developed relationships between a given non-convex quadratic programming problem QP and a derived unconstrained (but non-differentiable) quadratic problem NP. We have established that any local minimum of NP gives a corresponding local minimum of QP. Furthermore, the isolated local minimizers of both QP and NP are in one-to-one correspondence.

For the case that the Hessian of QP has exactly one negative eigenvalue, we have developed an algorithm to compute all isolated local minimizers and some non-isolated local minimizers of QP. In addition, the algorithm will compute the global minimizer of QP, provided it exists, and will provide the information that QP is unbounded from below when that is the case. The algorithm is illustrated by applying it to a problem from the literature and some variations of it.

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